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L^∞-MULTIVARIATE APPROXIMATION THEORY*

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1. Introduction. This paper is concerned with multivariate analogues of many standard results in the L^∞-approximation theory of functions of one real variable, cf. [12], [13], [14], [20], [21] and [24]. In particular, asymptotic bounds are given for the distance of a given smooth function from various finite-dimensional spaces such as multivariate trigonometric polynomials, multivariate algebraic polynomials and multivariate polynomial spline functions.

In § 2, we give a general result, Theorem 2.1, which enables us to deduce results about multivariate approximation theory in rectangular parallelepipeds from corresponding, single-variable results. In § 3, we discuss the convergence properties of multivariate Bernstein polynomials. In § 4, analogues of Jackson's theorems, cf. [21] and [24], are proved for multivariate trigonometric and polynomial approximation, the latter being for arbitrary compact subsets of R^N.

Multivariate polynomial spline approximation is discussed in § 5. Analogues of the fundamental results of deBoor [14] are formulated and proved for smooth, bounded domains in R^N. In § 6, error bounds are given for multivariate piecewise Lagrange interpolation. In § 7, a multivariate cubic spline interpolation scheme is introduced and analyzed. Finally, in § 8, applications of the results of § 5, are made to "h-asymptotically optimal approximation schemes" in Sobolev spaces and, in particular, to the Ritz method, in conjunction with multivariate polynomial spline functions, for approximating the solution of the Neumann problem for the Poisson equation for regular, bounded, open sets in R^N.

We end this section by recalling some multivariate notation which will be used throughout this paper.

For any point $x \equiv (x_1, \cdots, x_N) \in R^N$, $|x| \equiv (x_1^2 + \cdots + x_N^2)^{1/2}$. If $\alpha \equiv (\alpha_1, \cdots, \alpha_N)$ is an N-tuple with nonnegative integer components, then

$$x^\alpha \equiv x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \qquad D^\alpha \equiv D_1^{\alpha_1} \cdots D_N^{\alpha_N} \equiv \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}$$

and

$$|\alpha| \equiv \alpha_1 + \cdots + \alpha_N.$$

If Ω is a bounded set in R^N, $C(\Omega)$ denotes the set of real-valued continuous functions on Ω,

$$C^m(\Omega) \equiv \{f \in C(\bar{\Omega}) | D^\alpha f \in C(\bar{\Omega}) \text{ for all } |\alpha| \leqq m\}, \qquad m \geqq 1,$$

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$C^{m,\mu}(\Omega) \equiv \{f \in C^m(\Omega) \| D^\alpha f(x) - D^\alpha f(y)| \leqq K|x-y|^\mu$

for some positive constant K, for all $x, y \in \Omega$ and $|\alpha| = m\}$,

$$|f|_{m,\mu,\Omega} \equiv \sup_{|\alpha|=m} \sup_{x,y\in\Omega} |D^\alpha f(x) - D^\alpha f(y)|/|x-y|^\mu \quad \text{for all } f \in C^{m,\mu}(\Omega)$$

and

$$\|\cdot\|_\Omega \equiv |\cdot|_{0,0,\Omega}.$$

The Sobolev space $W^{m,p}(\Omega)$ denotes the space of $L^p(\Omega)$ real-valued functions which have weak derivatives in $L^p(\Omega)$ of order up to m,

$$\|f\|_{W^{m,p}(\Omega)} \equiv \left(\sum_{|\alpha| \leqq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p},$$

and $W_0^{m,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{W^{m,p}(\Omega)}$. If S is any measurable set in R^N and $\mathscr{M}$ is a bounded linear mapping of $L^\infty(S)$ into itself, then we define $\|f\|_S \equiv \|f\|_{L^\infty(S)}$ for all $f \in L^\infty(S)$ and $\|\mathscr{M}\| \equiv \sup_{f\neq 0} \|\mathscr{M}f\|_S/\|f\|_S$. Finally, the symbol K will be used repeatedly to denote a positive constant.

2. General results. For each positive integer, $1 \leqq i \leqq N$, let $-\infty < a_i < b_i < \infty$, $I_i \equiv [a_i, b_i]$, and L_i be a mapping of $C(I_i)$, the space of continuous real-valued functions on I_i, into $C(I_i)$ such that L_i is Lipschitz continuous with constant λ_i. If $H \equiv \bigtimes_{i=1}^N I_i \subset R^N$ and $f \in C(H)$, then $L_i f$ is interpreted to mean that L_i is applied to f viewed as a function of the ith variable, x_i, with the other variables x_j, $1 \leqq j \leqq N$, $j \neq i$ held fixed.

LEMMA 2.1. *If $g \in C(H)$, then $L_i g \in C(H)$ for $1 \leqq i \leqq N$.*

Proof. Let x and y be any two points in H. Then

$$\begin{aligned}
&|(L_i g)(x) - (L_i g)(y)| \\
&\leqq |(L_i g)(x_1, \cdots, x_i, \cdots, x_N) - (L_i g)(y_1, \cdots, x_i, \cdots, y_N)| \\
&\quad + |(L_i g)(y_1, \cdots, x_i, \cdots, y_N) - (L_i g)(y_1, \cdots, y_i, \cdots, y_n)| \\
&\leqq \|(L_i g)(x_1, \cdots, x, \cdots, x_N) - (L_i g)(y_1, \cdots, x, \cdots, y_N)\|_{I_i} \\
&\quad + |(L_i g)(y_1, \cdots, x_i, \cdots, y_N) - (L_i g)(y_1, \cdots, y_i, \cdots, y_N)| \\
&\leqq \lambda_i \|g(x_1, \cdots, x, \cdots, x_N) - g(y_1, \cdots, x, \cdots, y_N)\|_{I_i} \\
&\quad + |(L_i g)(y_1, \cdots, x_i, \cdots, y_N) - (L_i g)(y_1, \cdots, y_i, \cdots, y_N)|.
\end{aligned}$$

Hence, given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x - y\|_\infty \leqq \delta$ implies $|(L_i g)(x) - (L_i g)(y)| \leqq \varepsilon$. This completes the proof.

THEOREM 2.1. *If $f \in C(H)$, $L^k \equiv L_k L_{k-1} \cdots L_1$, $1 \leqq k \leqq N$, and $\varepsilon_i \equiv \|f - L_i f\|_H$, then*

$$\|f - L^N f\|_H \leqq \sum_{i=1}^N \left\{ \varepsilon_i \prod_{j=i+1}^N \lambda_j \right\}.$$

Proof. The result follows directly by induction on N and the observation that if $H_k \equiv \bigtimes_{\substack{i=1\\ i\neq k}}^{N} I_i$, then

$$\begin{aligned}\|f - L^k f\|_H &\leqq \|f - L_k f\|_H + \|L_k f - L_k L^{k-1} f\|_H \\ &= \|f - L_k f\|_H + \| \, \|L_k(f - L^{k-1}f)\|_{I_k}\|_{H_k} \\ &\leqq \|f - L_k f\|_H + \lambda_k \| \, \|f - L^{k-1}f\|_{I_k}\|_{H_k} \\ &\leqq \varepsilon_k + \lambda_k \|f - L^{k-1}f\|_H.\end{aligned}$$

3. Multivariate Bernstein polynomials. In this section we apply the result in § 2 to the study of the convergence properties of multivariate Bernstein polynomials. We start with the definition of the moduli of continuity.

DEFINITION 3.1. If $f \in C(H)$, define the *moduli of continuity*,

$$\omega_i(f; h) \equiv \max_{x \in H} \{|f(x_1, \cdots, x_i, \cdots, x_N) - f(x_1, \cdots, x_i', \cdots, x_N)| \; a_i \leqq x_i' \leqq b_i, |x_i - x_i'| \leqq h\}.$$

It is clear, from the uniform continuity of $f \in C(H)$, that $\lim_{h \to 0} \omega_i(f; h) = 0$.

We now consider the case in which $a_i = 0, b_i = 1, I_i \equiv I = [0, 1], 1 \leqq i \leqq N$, $H_I \equiv \bigtimes_{i=1}^N I_i$, and $L_{i,n} \equiv B_n$, $1 \leqq i \leqq N$, the Bernstein mapping; i.e.,

$$B_n(f)(x) \equiv \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1 - x)^{n-k}.$$

Then

$$L_n^N \equiv B_n^N \equiv \sum_{k_1=0}^{n} \cdots \sum_{k_N=0}^{n} \binom{n}{k_1} \cdots \binom{n}{k_N} f\left(\frac{k_1}{n}, \cdots, \frac{k_N}{n}\right) \cdot x_1^{k_1} \cdots x_n^{k_N} \cdot (1 - x_1)^{n-k_1} \cdots (1 - x_N)^{n-k_N}.$$

We can give error estimates for the approximation of a continuous function $f \in C(H_I)$ by the associated multivariate Bernstein polynomials.

THEOREM 3.1. (i) *If $f \in C(H_I)$, then*

$$\|f - B_n^N f\|_{H_I} \leqq \tfrac{5}{4}\left(\sum_{i=1}^{N} \omega_i\left(f; \frac{1}{n}\right)\right).$$

(ii) *Moreover, if $f \in C^1(H_I)$, then*

$$\|f - B_n^N f\|_{H_I} \leqq \frac{3}{4n^{1/2}}\left(\sum_{i=1}^{N} \omega_i\left(D_i f; \frac{1}{n^{1/2}}\right)\right).$$

Proof. It is easily verified that B_n is a linear mapping of $C[0, 1]$ into $C[0, 1]$ and $\|B_n\| = 1$. Hence, the sequence $\{B_n\}_{n=1}^{\infty}$ is uniformly Lipschitz continuous with constant $\lambda = 1$ and part (i) follows from Theorem 2.1 and a result of Popoviciu, cf. [20, Theorem 1.6.1, p. 20], which states that under the hypotheses of (i), $\varepsilon_i(n) \leqq \frac{5}{4}\omega_i(f; 1/n)$. Part (ii) follows from Theorem 2.1 and a result of Lorentz [20, Theorem 1.6.2, p. 21] which states that under the hypotheses of (ii),

$$\varepsilon_i(n) \leqq \frac{3}{4n^{1/2}}\omega_i\left(D_i f; \frac{1}{n^{1/2}}\right).$$

COROLLARY 1. *If $f \in C(H_I)$, then $B_n^N f$ converges uniformly to f as $n \to \infty$.*

We now obtain a multidimensional form of the Weierstrass approximation theorem.

COROLLARY 2. *If Ω is a closed, bounded subset of R^N and $f \in C(\Omega)$, then there exists a sequence of polynomials $\{p_n(x)\}_{n=1}^{\infty}$ such that $\{p_n(x)\}_{n=1}^{\infty}$ converges uniformly to f on Ω.*

Proof. Let H be any hypercube containing Ω. By the well-known Tietze extension theorem, there exists a $\tilde{f} \in C(H)$ such that $\tilde{f}(x) = f(x)$ for all $x \in \Omega$. Let $\{\tilde{B}_n^N \tilde{f}\}_{n=1}^{\infty}$ denote the sequence of multivariate Bernstein polynomials for $\tilde{f}$, normalized to the hypercube H. By Corollary 1, $\tilde{B}_n^N f$ converges uniformly to $\tilde{f}$ on H as $n \to \infty$ and hence $\tilde{B}_n^N \tilde{f}$ converges uniformly to f on Ω.

4. Multivariate trigonometric and polynomial approximation. In this section we prove multidimensional analogues of the well-known results of Jackson and Lebesgue on trigonometric and polynomial approximation. We start by recalling some key results of Jackson, cf. [12], [21] and [24].

In this section, let $I_\pi \equiv [-\pi, \pi]$, $C_{2\pi}$ denote the real-valued functions $f \in C^0(-\infty, \infty)$ which are periodic with period 2π, i.e., $f(x + 2\pi) = f(x)$ for all real x, and

$$L_{n,r}(s) \equiv \frac{1}{\lambda_{n,r}}\left(\frac{\sin(ns/2)}{\sin(s/2)}\right)^{2r},$$

$\lambda_{n,r}$ is determined by the condition that $\int_{-\pi}^{\pi} L_{n,r}(s)\,ds = 1$; r and n are positive integers. Letting $K_{n,r}(s) \equiv L_{m,r}(s)$ where $m \equiv [n/r] + 1$, we have that $K_{n,r}(s)$ is a trigonometric polynomial of degree n, i.e.,

$$K_{n,r}(s) \equiv A + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

Letting

$$J_{n,t}(f) \equiv -\int_{-\pi}^{\pi} K_{n,r}(s) \sum_{k=1}^{t+1} (-1)^k \binom{t+1}{k} f(x + ks)\,ds,$$

where r is fixed to be the smallest integer with $r \geqq \frac{1}{2}(t + 3)$, we have that $J_{n,t} f$ is a trigonometric polynomial of degree n and $J_{n,t}$ is a bounded mapping of $C_{2\pi}$ into $T_n \subset C_{2\pi}$, where T_n denotes the set of all real trigonometric polynomials of degree n.

In fact,

$$\|J_{n,t}\| \leqq \mu_t \equiv \max_{1 \leqq k \leqq t+1} \binom{t+1}{k} \geqq 1.$$

For each positive integer t and $1 \leqq p \leqq \infty$, let $K^{t,p}(-\infty, \infty)$ denote the set of functions $f \in C^{t-1}(-\infty, \infty)$ such that $D^{t-1}f$ is absolutely continuous and $D^t f \in L^p(-\infty, \infty)$. With $E(f, T_n) \equiv \inf\{\|f - t_n\|_{I_\pi} | t_n \in T_n\}$, Jackson has shown, cf. [12], [21] and [24], the following results.

THEOREM 4.1. (i) *There exists a positive constant K such that*

$$E(f, T_n) \leqq \|f - J_{n,t}f\|_{I_\pi} \leqq \frac{K}{n^t}\omega\left(D^t f; \frac{1}{n}\right), \qquad n \geqq 1,$$

for all $f \in C_{2\pi} \cap C^t(-\infty, \infty)$.

(ii) *There exists a positive constant K such that*

$$E(f, T_n) \leqq \|f - J_{n,t}f\|_{I_\pi} \leqq \frac{K}{n^{t+\alpha}}|f|_{t,\alpha,I_\pi}, \qquad n \geqq 1,$$

for all $f \in C_{2\pi} \cap C^{t,\alpha}(-\infty, \infty)$.

(iii) *There is a positive constant K such that*

$$E(f, T_n) \leqq \|f - J_{n,t}f\|_{I_\pi} \leqq \frac{K}{n^{t+1}}\|D^{t+1}f\|_{I_\pi}, \qquad n \geqq 1,$$

for all $f \in C_{2\pi} \cap K^{t+1,\infty}(-\infty, \infty)$.

For every $f \in C_{2\pi}$ let $S_n(f)$ be a partial sum of its Fourier series, i.e.,

$$S_n(f)(x) \equiv A + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$

where

$$A \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt, \qquad a_k \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt \quad \text{and} \quad b_k \equiv \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt.$$

Then we have the following well-known result, cf. [24].

THEOREM 4.2. (i) $\|S_n\| \leqq 2\ln n$ *for all* $n \geqq 1$.

(ii) *There exists a positive constant K such that*

$$\|S_n(f) - f\|_{I_\pi} \leqq (3 + \ln n)E(f, T_n) \leqq (3 + \ln n)K\omega\left(f; \frac{1}{n}\right), \qquad n \geqq 1,$$

for all $f \in C_{2\pi}$.

(iii) *If*

$$\lim_{\delta \to 0}[\omega(f;\delta)\ln\delta] = 0 \qquad \text{for } f \in C_{2\pi},$$

then f can be expanded in a uniformly convergent Fourier series.

Part (ii) is due to Lebesque, and part (iii) is known as the Dini–Lipschitz condition, which is fulfilled by all functions belonging to $C^{0,\gamma}(R)$ for any $\gamma > 0$.

Proceeding to the multivariate results, we let $J_{n,t}^N \equiv J_{n,t,N} \cdots J_{n,t,1}$, $S_n^N \equiv S_{n,N} \cdots S_{n,1}$, $H_\pi \equiv \times_{i=1}^N [-\pi, \pi]_i$, $C_{2\pi}^N$ be the set of all real-valued functions $f \in C^0(R^N)$ which are periodic with respect to each variable with period 2π, T_n^N denote the set of all real trigonometric polynomials, $t_n(x_1, \cdots, x_N)$, in N-variables, of degree n in each variable, and $E(f; T^N) \equiv \inf\{\|f - t_n\|_{H_\pi} | t_n \in T_n^N\}$.

For each positive integer t and $1 \leqq p \leqq \infty$ let $K^{t,p}(R^N)$ denote the set of functions $f \in C(R^N)$ such that for each $1 \leqq i \leqq N$, $D_i^{t-1} f$ is absolutely continuous and $D_i^t f \in L^p(R^N)$. Combining the results of Theorems 2.1 and 4.1, we obtain Theorem 4.3.

THEOREM 4.3. (i) *There is a positive constant K such that*

$$E(f, T_n^N) \leqq \|f - J_{n,t}^N f\|_{H_\pi} \leqq \frac{K}{n^t}\left(\sum_{i=1}^N \mu_t^{N-i} \omega_i\left(D_i^t f; \frac{1}{n}\right)\right), \qquad n \geqq 1,$$

for all $f \in C_{2\pi}^N \cap C^t(R^N)$.

(ii) *There is a positive constant K such that*

$$E(f; T_n^N) \leqq \|f - J_{n,t}^N f\|_{H_\pi} \leqq \frac{K}{n^{t+\gamma}}\left(\sum_{i=1}^N \mu_t^{N-i}\right) |f|_{t,\gamma,H_\pi}, \qquad n \geqq 1,$$

for all $f \in C_{2\pi}^N \cap C^{t,\gamma}(R^N)$.

(iii) *There is a positive constant K such that*

$$E(f; T_n^N) \leqq \|f - J_{n,t}^N f\|_{H_\pi} \leqq \frac{K}{n^{t+1}}\left(\sum_{i=1}^N \mu_t^{N-i} \|D_i^{t+1} f\|_{H_\pi}\right), \qquad n \geqq 1,$$

for all $f \in C_{2\pi}^N \cap K^{t+1,\infty}(R^N)$.

Combining Theorems 2.1, 4.2 and 4.3 we obtain the following theorem.

THEOREM 4.4. (i) *There exists a positive constant K such that*

$$\begin{aligned} \|S_n^N(f) - f\|_{H_\pi} &\leqq (3 + \ln n) \sum_{i=1}^N \{E(f, T_{n,i})(2 \ln n)^{N-i}\} \\ &\leqq (3 + \ln n)(2 \ln n)^{N-1}\left(\sum_{i=1}^N E(f, T_{n,i})\right) \\ &\leqq (3 + \ln n)(2 \ln n)^{N-1} K\left(\sum_{i=1}^N \omega_i\left(f; \frac{1}{n}\right)\right) \end{aligned}$$

for all $f \in C_{2\pi}^N$.

(ii) *If*

$$\lim_{\delta \to 0} [\omega_i(f; \delta)(\ln \delta)^N] = 0$$

for all $1 \leqq i \leqq N$ *for* $f \in C_{2\pi}^N$, *then f can be expanded in a uniformly convergent Fourier series in N variables.*

We remark that the condition given in (ii) is satisfied for all functions $f \in C^{0,\gamma}(R^N)$ for any $\gamma > 0$.

By a standard change of variable argument, cf. [21], [24], we can use the results of Theorem 4.3 to deduce results about polynomial approximation. Let P_n^N denote the set of all polynomials $p_n(x_1, \cdots, x_N)$ in N variables, of degree n in each variable. If H is any rectangular parallelepiped as in § 2 and $f \in C(H)$, we define

$$E(f; P_n^N) \equiv \inf\{\|f - p_n\|_H | p_n \in P_n^N\}$$

and, for each positive integer t and $1 \leqq p \leqq \infty$, $K^{t,p}(H)$ to be the set of functions $f \in C(H)$ such that for each $1 \leqq i \leqq N$, $D_i^{t-1}f$ is absolutely continuous and $D_i^t f \in L^p(H)$.

THEOREM 4.5. (i) *There exists a positive constant K such that*

$$E(f; P_n^N) \leqq \frac{K}{n^t}\left(\sum_{i=1}^N \omega_i\left(D_i^t f; \frac{1}{n}\right)\right), \qquad n \geqq 1,$$

for all $f \in C^t(H)$.

(ii) *There exists a positive constant K such that*

$$E(f; P_n^N) \leqq \frac{K}{n^{t+\gamma}} |f|_{t,\gamma,H}, \qquad n \geqq 1,$$

for all $f \in C^{t,\gamma}(H)$.

(iii) *There exists a positive constant K such that*

$$E(f; P_n^N) \leqq \frac{K}{n^{t+1}}, \qquad n \geqq 1,$$

for all $f \in C^t(H) \cap K^{t+1,\infty}(H)$.

In order to prove an analogue of Theorem 4.5 for arbitrary compact subsets of R^N, we must consider a fundamental result of Whitney [28]. Let S be a closed subset of R^N. A function defined on S is said to be of class $C_*^m(S)$ if and only if there exist functions $\{f_k(x)\}_{|k| \leqq m}$ defined on S such that for every point $p \in S$ and every positive constant ε there is a neighborhood $N_{p,\varepsilon}$ of p such that for every pair of points x, y in $N_{p,\varepsilon} \cap S$,

$$|f_k(x) - D^k p_m(x, y)| \leqq \varepsilon r^{m-|k|},$$

where

$$p_m(x, y) \equiv \sum_{|k| \leqq m} \frac{f_k(y)}{k!}(x - y)^k$$

and r is the distance between the points x and y. We remark that if $S = R^N$, then $f \in C_*^m(R^N)$ if and only if $f \in C^m(R^N)$.

THEOREM 4.6. *If S is a closed subset of R^N and $f \in C_*^m(S)$ for some $m \geqq 0$, then there exists an $\tilde{f} \in C^m(R^N)$ such that $\tilde{f} \in C^\infty(R^N - S)$ and $\tilde{f}(x) = f(x)$ for all $x \in S$.*

In the case of $m = 0$, the following stronger result has been proved by Hestenes [18].

THEOREM 4.7. *If S is a closed subset of R^N and $f \in C(S)$, then there exists an $\tilde{f} \in C(R^N)$ such that $f(x) \equiv \tilde{f}(x)$ for all $x \in S$ and $\omega_S(f;h) = \omega_{R^N}(f;h)$ for all $h > 0$, where*

$$\omega_S(f;h) \equiv \sup\{|f((x) - f(y)| \,|\, x, y \in S \quad \text{and} \quad \|x - y\|_\infty \leqq h\}$$

and

$$\omega_{R^N}(\tilde{f};h) \equiv \sup\{|\tilde{f}(x) - \tilde{f}(y)| \,|\, x, y \in R^N \quad \text{and} \quad \|x - y\|_\infty \leqq h\}.$$

Combining the results of Theorems 4.5 and 4.6 we obtain the following theorem.

THEOREM 4.8. *Let S be a compact subset of R^N.*

(i) *There exists a positive constant K such that*

$$E(f;P_n^N) \equiv \inf\{\|f - p_n\|_S | p_n \in P_n^N\} \leqq K\omega_S\left(f;\frac{1}{n}\right), \quad n \geqq 1,$$

for all $f \in C^0(S)$.

(ii) *If $f \in C_*^m(S)$, $m \geqq 1$, then there exists a positive constant $K(f)$, depending on f, such that $E(f;P_n^N) \leqq K(f)/n^t$, $n \geqq 1$.*

Proof. We prove (ii); the proof of (i) is essentially identical except that Theorem 4.7 is used in place of Theorem 4.6. By Theorem 4.6, f can be extended to a function $\tilde{f} \in C^m(R^N)$. We consider the restriction of $\tilde{f}$ to some rectangular parallelepiped, H, which contains S. Then, since

$$\inf\{\|f - p_n\|_S | p_n \in P_n^N\} \leqq \inf\{\|\tilde{f} - p_n\|_H | p_n \in P_n^N\},$$

it suffices to consider polynomial approximations of $\tilde{f}$ over K. The result then follows directly from Theorem 4.5.

In order to improve the above result we need to make further assumptions about the set S. Following [23] we have the following definition.

DEFINITION 4.1. A bounded open set $\Omega \subset R^N$ is said to be *regular* if and only if there exists a finite open covering $\{u_i\}_{i=1}^n$ of the boundary $\partial\Omega$, finitely many finite cones $\{\gamma_i\}_{i=1}^n$, and a positive number ε such that every point of $\partial\Omega$ is the center of a sphere of radius ε entirely contained in one of the sets u_i and every point of $u_i \cap \Omega$ is the vertex of a translate of γ_i, $1 \leqq i \leqq n$, contained entirely in Ω.

We remark that a bounded open set is regular if and only if it is strongly Lipschitz in the sense of Morrey [23] and every bounded, open, convex set is regular [23, Lemma 3.4.1]. In this context Calderon has proved the following extension theorem, cf. [8] and [23, Theorem 3.4.3].

THEOREM 4.9. *Let Ω be a regular, bounded open set in R^N and $\bar{\Omega} \subset \operatorname{int} S$, where S is a bounded measurable set in R^N.*

(i) *If $1 < p < \infty$ and m is a nonnegative integer, then there is a bounded linear extension mapping $\varepsilon: W^{m,p}(\Omega) \to W_0^{m,p}(S)$ such that $\varepsilon u(x) = u(x)$, $x \in \Omega$, for all $u \in W^{m,p}(\Omega)$.*

(ii) *If $0 \leqq \gamma \leqq 1$ and m is a nonnegative integer, then there is a bounded linear extension mapping $\varepsilon: C^{m,\gamma}(\bar{\Omega}) \to C^{m,\gamma}(\bar{S})$ such that $\varepsilon u(x) = u(x)$, $x \in \Omega$, for all $u \in C^{m,\gamma}(\bar{\Omega})$.*

Combining Theorems 4.5 and 4.9 we obtain the following theorem.

THEOREM 4.10. *If Ω is a regular, bounded, open set in R^N, then there exists a positive constant K such that*

$$E(f; P_n^N) \equiv \inf\{\|f - p_n\|_{\bar\Omega} | p_n \in P_n^N\} \leqq \frac{K}{n^{t+\gamma}} |f|_{t,\gamma,\bar\Omega}, \qquad n \geqq 1,$$

for all $f \in C^{t,\gamma}(\bar\Omega)$, t any nonnegative integer and $0 \leqq \gamma \leqq 1$.

For many applications of approximation theory results, the natural hypothesis on the smoothness of f is that f belongs to some Sobolev space, $W^{m,p}(\Omega)$. To prove an analogue of Theorem 4.10 with this type of hypothesis we need a special form of Sobolev's theorem which we now state for completeness, cf. [8] and [23].

THEOREM 4.11. *Let Ω be a regular, bounded, open set in R^N and $u \in W^{m,p}(\Omega)$, $1 < p < \infty$. If $m - N/p = k + \gamma$, where k is a nonnegative integer and $0 < \gamma < 1$, then there is a function $\bar u \in C^{k,\gamma}(\bar\Omega)$ such that $u = \bar u$ almost everywhere in Ω and there exists a positive constant K, independent of u, such that $\|\bar u\|_{k,\gamma,\bar\Omega} \leqq K\|u\|_{W^{m,p}(\Omega)}$. If $m - N/p = k + 1$, where k is a nonnegative integer, then for each $0 < \gamma < 1$ there is a function $\bar u_\gamma \in C^{k,\gamma}(\bar\Omega)$ and a positive constant K_γ, such that $u = \bar u_\gamma$ almost everywhere in Ω and $\|\bar u_\gamma\|_{k,\gamma,\bar\Omega} < K_\gamma\|u\|_{W^{m,p}(\Omega)}$.*

Combining the results of Theorems 4.10 and 4.11 we have the following theorem.

THEOREM 4.12. *Let Ω be a regular, bounded, open set in R^N. If $m - N/p = k + \gamma$, where k is a nonnegative integer, and $0 < \gamma < 1$, then there is a positive constant K such that*

$$E(f; P_n^N) \leqq \frac{K}{n^{k+\gamma}} \|f\|_{W^{m,p}(\Omega)}, \qquad n \geqq 1,$$

for all $f \in W^{m,p}(\Omega)$. If $m - N/p = k + 1$, where k is a nonnegative integer, then for each $0 < \gamma < 1$ there is a positive constant K_γ, such that

$$E(f; P_n^N) \leqq \frac{K_\gamma}{n^{k+\gamma}} \|f\|_{W^{m,p}(\Omega)}, \qquad n \geqq 1,$$

for all $f \in W^{m,p}(\Omega)$.

5. Multivariate polynomial spline approximation. In this section we generalize the recent, important results of deBoor [14] to multivariate polynomial spline approximation of functions defined on subsets of R^N. We start by considering the single variable case.

Let $I \equiv [a, b]$, $-\infty < a < b < \infty$, $\Delta : a = x_0 < x_1 < \cdots < x_m < x_{m+1} = b$, m a positive integer, be a partition of I, k be a positive integer, $z = (z_1, \cdots, z_m)$ the incidence vector associated with Δ be an m-vector with positive integer components such that $1 \leqq z_i \leqq k$, $1 \leqq i \leqq m$, and $S^k_{\Delta,z} \equiv S^k_{\Delta,z}(I)$ be the set of all polynomial spline functions of degree k on I with respect to Δ and z, i.e., $s(x) \in S^k_{\Delta,z}$ if and only if $s(x)$ is a polynomial of degree k on each subinterval $[x_i, x_{i+1}]$, $0 \leqq i \leqq m$, defined by Δ and $s(x)$ is $(k - z_i)$ times continuously differentiable at x_i, $1 \leqq i \leqq m$. We remark that in the special case of $k = 3$ and $z_i = 1$, $1 \leqq i \leqq m$, $S^k_{\Delta,z}$ is the space of classical cubic spline functions cf. [13], in the case of $k = 3$ and $z_i = 2, 1 \leqq i \leqq m$, $S^k_{\Delta,z}$ is the space of piecewise cubic Hermite polynomials, and in the case of $k = 3$ and $z_i = 3$, $1 \leqq i \leqq m$, $S^k_{\Delta,z}$ is the space of nonsmooth piecewise cubic Hermite polynomials.

For each positive integer t and $1 \leqq p \leqq \infty$, let $K^{t,p}(I)$ denote the set of functions $f \in C^{t-1}(I)$ such that $D^{t-1}f$ is absolutely continuous and $D^t f \in L^p(I)$. DeBoor [14] has proved the following important result.

THEOREM 5.1. *There exists a linear projector $P_{\Delta,k,z}$ of $C(I)$ onto $S^k_{\Delta,z}$ such that*

(i) *there exist positive constants M_k such that $\|P_{\Delta,k,z}\| \leqq M_k$,*

(ii) $\|f - P_{\Delta,k,z}f\|_I \leqq kM_k\omega(\bar{\Delta}; f)$ *for all $f \in C(I)$, where*

$$\bar{\Delta} \equiv \max_{0 \leqq i \leqq m}(\chi_{i+1} - \chi_i),$$

(iii) $\|f - P_{\Delta,k,z}f\|_I \leqq \hat{M}_k\hat{M}_{k-1} \cdots \hat{M}_{k-t}(\bar{\Delta})^t\omega(\bar{\Delta}; D^t f)$ *for all $f \in C^t(I)$, for all $1 \leqq t \leqq k$, where $\hat{M}_k \equiv kM_k$, $\hat{M}_0 = 1$ and*

(iv) *there exists a positive constant K such that* $\|f - P_{\Delta,k,z}f\|_I \leqq K(\bar{\Delta})^{t+1}\|D^{t+1}f\|_I$ *for all $f \in K^{t+1,\infty}(I)$, $0 \leqq t \leqq k$.*

For many applications, cf. § 8, it is important to have bounds for the norm of derivatives of the quantities $(f - P_{\Delta,k,z}f)$. Such bounds for the uniform norm are given by the following new result. We remark that if the components of the incidence vector z are large enough so that $D^j(f - P_{\Delta,k,z}f) \notin C(I)$, then $\|D^j(f - P_{\Delta,k,z}f)\|_I$ is interpreted to be the uniform norm over the set $I - \Delta$.

COROLLARY 1. (i) *If $(\bar{\Delta}/\Delta) \leqq \tau$, where $\Delta \equiv \min_{0 \leqq i \leqq m}(x_{i+1} - x_i)$, then*

$$\|D^j(f - P_{\Delta,k,z}f)\|_I \leqq B_{k,t,j,\tau}(\bar{\Delta})^{t-j}\omega(\bar{\Delta}; D^t f), \qquad 1 \leqq j \leqq t, \tag{5.1}$$

where

$$B_{k,t,j,\tau} \equiv \left[\frac{2^j\tau^j k^2(k^2 - 1^2) \cdots (k^2 - [j-1]^2)}{1 \cdot 3 \cdots (2j-1)}(1 + \hat{M}_k \cdots \hat{M}_{k-t+1}) + 1\right]\hat{M}_{k-t},$$

for all $f \in C^t(I)$, $1 \leqq t \leqq k$.

(ii) *If $(\bar{\Delta}/\Delta) \leqq \tau$, then there exists a positive constant K such that*

$$\|D^j(f - P_{\Delta,k,z}f)\|_I \leqq K\tau^j(\bar{\Delta})^{t-j+1}\|D^{t+1}f\|_I, \qquad 1 \leqq j \leqq t,$$

for all $f \in K^{t+1,\infty}(I)$, $1 \leqq t \leqq k$.

Proof. We prove only (i); the proof of (ii) is essentially identical. Set $q_0(x) \equiv P_{\Delta,k-t,1}(D^t f)(x)$, where

$$P_{\Delta,0,1}(D^k f)(x) \equiv D^k f\left(\frac{x_i + x_{i+1}}{2}\right)$$

for $x \in [x_i, x_{i+1}]$, $0 \leqq i \leqq m$. If $t = k$, then

$$\|q_0 - D^k f\|_I \leqq \omega(\Delta; D^k f) = \hat{M}_0\omega(\Delta; D^k f), \tag{5.2}$$

and if $t < k$, Theorem 5.1 yields

$$\|q_0 - D^t f\|_I \leqq \hat{M}_{k-t}\omega(\Delta; D^t f). \tag{5.3}$$

Define $q_j(x)$, $1 \leqq j \leqq t$, recursively by

$$q_j(x) \equiv D^{t-j}f(x_i) + \int_{x_i}^{x} q_{j-1}(t)\,dt,$$

$x \in [x_i, x_{i+1}]$, $0 \leqq i \leqq m$. Then $q_j(x)$ is a piecewise polynomial of degree $k - t + j$ on I, with respect to Δ, $Dq_j = q_{j-1}$, and

$$\|q_j - D^{t-j}f\|_I \leqq \bar{\Delta}\|q_{j-1} - D^{t-j+1}f\|_I, \qquad 1 \leqq j \leqq t. \tag{5.4}$$

In fact, if $x_* \in [x_i, x_{i+1}]$ is such that

$$|q_j(x_*) - D^{t-j}f(x_*)| = \|q_j - D^{t-j}f\|_I,$$

then

$$|q_j(x_*) - D^{t-j}f(x_*)| \leqq \int_{x_i}^{x_*} |q_{j-1}(t) - D^{t-j+1}f(t)|\,dt.$$

Inequalities (5.2), (5.3) and (5.4) yield the result that

$$\|D^j(q_t - f)\|_I \leqq \hat{M}_{k-t}(\bar{\Delta})^{t-j}\omega(\bar{\Delta}; D^t f), \qquad 0 \leqq j \leqq t. \tag{5.5}$$

By (5.5) and (iii) of Theorem 5.1, we have that

$$\|q_t - P_{\Delta,k,z}f\|_I \leqq (1 + \hat{M}_k \cdots \hat{M}_{k-t+1})\hat{M}_{k-t}(\bar{\Delta})^t\omega(\bar{\Delta}; D^t f).$$

Hence, by a Markov type result, cf. [27, Theorem 3.1.8, p. 138], since $q_t - P_{\Delta,k,z}f$ is a piecewise polynomial of degree k with respect to Δ,

$$\|D^j(q_t - P_{\Delta,k,z}f)\|_I \leqq \left(\frac{2}{\underline{\Delta}}\right)^j \frac{k^2(k^2 - 1^2)\cdots(k^2 - [j-1]^2)}{1\cdot 3\cdots(2j-1)} \cdot (1 + \hat{M}_k \cdots \hat{M}_{k-t+1})\hat{M}_{k-t}(\bar{\Delta})^t\omega(\Delta; D^t f), \quad 0 \leqq j \leqq t. \tag{5.6}$$

The required result follows from (5.5), (5.6) and the triangle inequality. This completes the proof.

If $f(a) = f(b) = 0$, we may wish to have a linear projector $Q_{\Delta,k,z}$ with the property that $Q_{\Delta,k,z}f(a) = Q_{\Delta,k,z}f(b) = 0$, cf. § 7. Such a projector is given by

$$Q_{\Delta,k,z}f(x) \equiv P_{\Delta,k,z}f(x) - \frac{b-x}{b-a}P_{\Delta,k,z}f(a) - \frac{x-a}{b-a}P_{\Delta,k,z}f(b).$$

From this definition, Theorem 5.1 and Corollary 1, we have the following new result.

COROLLARY 2. (i) $\|Q_{\Delta,k,z}\| \leqq 2M_k$.

(ii) $\|f - Q_{\Delta,k,z}f\|_I \leqq 2kM_k\omega(\bar{\Delta}; f)$ *for all* $f \in C_0(I)$, *i.e.*, $f \in C(I)$ *and* $f(a) = f(b) = 0$.

(iii) *If* $(\bar{\Delta}/\underline{\Delta}) \leqq \tau$, *then*

$$\|f - Q_{\Delta,k,z}f\|_I \leqq 2\hat{M}_k \cdots \hat{M}_{k-t}(\bar{\Delta})^t\omega(\bar{\Delta}; D^t f), \tag{5.7}$$

$$\|D(f - Q_{\Delta,k,z}f)\|_I \leqq \left(B_{k,t,1,\tau} + \frac{2}{b-a}\hat{M}_k \cdots \hat{M}_{k-t}\bar{\Delta}\right)(\bar{\Delta})^{t-1}\omega(\bar{\Delta}; D^t f), \tag{5.8}$$

and

$$\|D^j(f - Q_{\Delta,k,z}f)\|_I \leqq B_{k,t,j,\tau}(\bar{\Delta})^{t-j}\omega(\bar{\Delta}; D^t f), \tag{5.9}$$

$2 \leqq j \leqq t$, *for all* $f \in C^t(I) \cap C_0(I)$ *for some* $1 \leqq t \leqq k$.

(iv) *If* $(\bar{\Delta}/\underline{\Delta}) \leqq \tau$, *then there exists a positive constant* K *such that*

$$\|D^j(f - Q_{\Delta,k,z}f)\|_I \leqq K\tau^j(\bar{\Delta})^{k-j+1}\|D^{k+1}f\|_I, \qquad 1 \leqq j \leqq k,$$

for all $f \in C_0(I) \cap K^{k+1,\infty}(I)$.

We now proceed to the multivariate results. As in § 2, let $I_i \equiv [a_i, b_i]$, $-\infty < a_i < b_i < \infty$, $1 \leqq i \leqq N$, $H \equiv \bigtimes_{i=1}^N I_i$, Δ_i, $1 \leqq i \leqq N$, denote a partition of $[a_i, b_i]$, and $\rho \equiv \bigtimes_{i=1}^N \Delta_i$. Given $f \in C(H)$, for each $1 \leqq i \leqq N$, define $P_i(f) \equiv P_{\Delta_i,k,z(i)}f$ as the function obtained by applying the mapping $P_{\Delta_i,k,z(i)}$ to $f(x)$, viewed as a function of x_i, while holding the variables x_k, $k \neq i$, fixed, and $P^N \equiv P^N_{f,k} \equiv P_N P_{N-1} \cdots P_1$.

Combining Theorems 2.1 and 5.1, we have the following new result.

THEOREM 5.2. (i) $\|P^N\| \leqq (M_k)^N$.

(ii) $\|f - P^N f\|_H \leqq \hat{M}_k \sum_{i=1}^N \omega_i(\bar{\Delta}_i; f)(M_k)^{N-i}$ *for all* $f \in C(H)$.

(iii) $\|f - P^N f\|_H \leqq \hat{M}_k \cdots \hat{M}_{k-t} \sum_{i=1}^N (\bar{\Delta}_i)^t \omega_i(\bar{\Delta}_i; D_i^t f)(M_k)^{N-i}$ *for all* $f \in C(H)$ *such that* $D_i^t f \in C(H)$, $1 \leqq i \leqq N$, *for some* $1 \leqq t \leqq k$.

(iv) *There exists a positive constant* K *such that*

$$\|f - P^N f\|_H \leqq K \sum_{i=1}^N (\bar{\Delta}_i)^{t+1}\|D_i^{t+1}f\|_I$$

for all $f \in K^{t+1,\infty}(H)$, $0 \leqq t \leqq k$.

Given $f \in C_0(H)$ i.e., $f \in C(H)$ and $f(x) = 0$ for all $x \in \partial K$, for each $1 \leqq i \leqq N$, define $Q_i(f) \equiv Q_{\Delta_i,k,z(i)}f$ as the function obtained by applying the mapping $Q_{\Delta_i,k,z(i)}$ to $f(x)$, viewed as a function of x_i, while holding the variables x_k, $k \neq i$, fixed, and $Q^N \equiv Q^N_{\rho,k} \equiv Q_N Q_{N-1} \cdots Q_1$. Combining Theorem 2.1 and Corollary 2 of Theorem 5.1, we have the following new result.

THEOREM 5.3. (i) $\|Q^N\| \leqq (2M_k)^N$.

(ii) $\|f - Q^N f\|_H \leqq 2\hat{M}_k \sum_{i=1}^N \omega_i(\bar{\Delta}_i; f)(2\hat{M}_k)^{N-i}$ *for all* $f \in C_0(H)$.

(iii) $\|f - Q^N f\|_H \leqq 2\hat{M}_k \cdots \hat{M}_{k-r} \sum_{i=1}^N (\bar{\Delta}_i)^t \omega_i(\bar{\Delta}_i; D_i^t f)(2M_k)^{N-i}$ *for all* $f \in C_0(H)$ *such that* $D_i^t f \in C(H)$, $1 \leqq i \leqq N$, *for some* $1 \leqq t \leqq k$.

(iv) *There exists a positive constant* K *such that for all* $f \in C_0(H) \cap K^{t+1,\infty}(H)$,

$$\|f - Q^N f\|_H \leqq K \sum_{i=1}^N (\bar{\Delta}_i)^{t+1}\|D_i^{t+1}f\|_H$$

for all $0 \leqq t \leqq k$.

For many applications we are interested in bounds for the norms of partial derivatives of the quantities $f - P^N f$ and $f - Q^N f$. We start with the following important basic new results.

LEMMA 5.1. *If* $f \in C(H)$ *is such that* $D_i f \in C(H)$ *for some* $1 \leqq i \leqq N$, *then* $D_i(P_j f) = P_j(D_i f)$ *for all* $1 \leqq j \leqq N$ *such that* $j \neq i$.

Proof. By an important result of deBoor [14],

$$P_j f \equiv \sum_{l=1}^{q} \lambda_l(f)\varphi_l,$$

where the λ_l are continuous linear functionals on $C[a_j, b_j]$ and $\{\varphi_l\}_{l=1}^q$ is a linearly independent basis for $S_{\Delta_j,k,z(j)}$. By the well-known Riesz representation theorem, there exist $\{\alpha_l(x_j)\}_{l=1}^q$, functions of bounded variation, such that

$$P_j f(x_j) = \sum_{l=1}^{q} \varphi_l(x_j) \int_{a_j}^{b_j} f(x_1, \cdots, x_j, \cdots, x_N)\, d\alpha_l(x_j).$$

The result then follows from the well-known result concerning the commutation of differentiation and Stieltjes integration.

In similar fashion we have the following lemmas.

LEMMA 5.2. *If $f \in C_0(H)$ is such that $D_i f \in C(H)$ for some $1 \leqq i \leqq N$, then $D_i(Q_j f) = Q_j(D_i f)$ for all $1 \leqq j \leqq N$ such that $j \neq i$.*

LEMMA 5.3. *If $f \in C(H)$, $P_iP_jf = P_jP_if$, $1 \leqq i, j \leqq N$, and if $f \in C_0(H)$, then $Q_iQ_jf = Q_jQ_if$, $1 \leqq i, j \leqq N$.*

DEFINITION 5.1. A collection C of partitions $\rho \equiv \bigtimes_{i=1}^N \Delta_i$ of H is said to be *quasi-uniform* if and only if there exists a positive constant τ such that $(\bar{\Delta}_i/\underline{\Delta}_j) \leqq \tau$, $1 \leqq i, j \leqq m$.

Using Lemmas 5.1, 5.2 and 5.3 with Theorem 5.2, we obtain a new result. We remark that if $D_l^j P^N f \notin C(H)$, $\|D_l^j(f - P^N f)\|_H$ is interpreted as the uniform norm over the interior of the cells defined by the partition.

THEOREM 5.4. *Let C be a quasi-uniform collection of partitions ρ of H.*

(i) *There exists a positive constant K such that*

$$\|D_l^j(f - P^N f)\|_H \leqq K \sum_{i=1}^{N} (\bar{\Delta}_i)^{t-j} \omega_i(\bar{\Delta}_i; D_i^t f), \tag{5.10}$$

$1 \leqq j \leqq t$, $1 \leqq l \leqq N$, for all $1 \leqq t \leqq k$, $f \in C(H)$ such that $D_i^t f \in C(H)$, $1 \leqq i \leqq N$, and all $\rho \in C$.

(ii) *There exists a positive constant K such that*

$$\|D_l^j(f - P^N f)\|_H \leqq K \sum_{i=1}^{N} (\bar{\Delta}_i)^{t-j+1} \|D_i^{t+1} f\|_H, \quad 1 \leqq j \leqq t, \quad 1 \leqq l \leqq N, \tag{5.11}$$

for all $f \in K^{t+1,\infty}(H)$, $1 \leqq t \leqq k$, and all $\rho \in C$.

Proof. There exists a positive constant M such that

$$\begin{aligned} \|D_l^j(f - P^N f)\| &\leqq \|D_l^j(f - P_l f)\|_H + \left\| D_l^j P_l \left(f - \prod_{\substack{i=1 \\ i \neq l}}^{N} P_i f \right) \right\|_H \\ &\leqq \|D_l^j(f - P_l f)\|_H + \frac{M}{\underline{\Delta}_l^j} \left\| f - \prod_{\substack{i=1 \\ i \neq l}}^{N} P_i f \right\|_H, \end{aligned}$$

where we have used Markov's inequality, cf. [27, Theorem 3.1.8, p. 138], and (i) of Theorem 5.1. The result then follows from Theorem 5.2 and Corollary 1 of Theorem 5.1.

Analogous to the above result we have the following theorem.

THEOREM 5.5. *Let C be a quasi-uniform collection of partition of H.*

(i) *There exists a positive constant K such that*

$$\|D_l^j(f - Q^N f)\|_H \leqq K \sum_{i=1}^{N} (\bar{\Delta}_i)^{t-j} \omega_i(\bar{\Delta}_i ; D_i^t f), \qquad 1 \leqq j \leqq t, \tag{5.12}$$

$1 \leqq l \leqq N$, *for all $f \in C_0(H)$ such that $D_i^t f \in C(H)$, $1 \leqq i \leqq N$, $1 \leqq t \leqq k$, and all $\rho \in C$.*

(ii) *There exists a positive constant K such that*

$$\|D_l^j(f - Q^N f)\|_H \leqq K \sum_{i=1}^{N} (\bar{\Delta}_i)^{t-j+1} \|D_i^{t+1} f\|_H, \qquad 1 \leqq j \leqq t, \tag{5.13}$$

$1 \leqq l \leqq N$, *for all $f \in C_0(H) \cap K^{t+1,\infty}(H)$, $0 \leqq t \leqq k$, and all $\rho \in C$.*

We now wish to state analogues of Theorems 5.4 and 5.5 for mixed partial derivatives. The proofs are essentially the same as the proof of Theorem 5.4 with the added information of Lemmas 5.1, 5.2 and 5.3. For each positive integer t and $1 \leqq p \leqq \infty$, let $S^{t,p}(H)$ denote the set of functions $f \in C^{t-1}(H)$ such that $D^\alpha f$ is absolutely continuous in H for all $|\alpha| = t - 1$ and $D^\alpha f \in L^p(H)$ for all $|\alpha| = t$.

THEOREM 5.6. *Let C be a quasi-uniform collection of partitions of H.*

(i) *There exists a positive constant K such that*

$$\|D^\alpha(f - P^N f)\|_H \leqq K(\bar{\rho})^{t-|\alpha|} \omega^{(t)}(\bar{\rho}, f), \qquad 0 \leqq |\alpha| \leqq t,$$

where $\bar{\rho} \equiv \max_{1 \leqq i \leqq N} \bar{\Delta}_i$ and $\omega^{(t)}(\bar{\rho}; f) \equiv \max_{|q|=t} \omega(\bar{\rho}; D^q f)$, for all $f \in C^t(H)$, $0 \leqq t \leqq k$, and all $\rho \in C$.

(ii) *There exists a positive constant K such that*

$$\|D^\alpha(f - P^N f)\|_H \leqq K(\bar{\rho})^{t-|\alpha|+1}\big(\max_{|q|=t+1} \|D^q f\|_H\big), \quad 0 \leqq |\alpha| \leqq t \leqq k,$$

for all $f \in S^{t+1,\infty}(H)$, $0 \leqq t \leqq k$, and all $\rho \in C$.

As a direct corollary of Theorem 5.6 we have the following theorem.

THEOREM 5.7. *If C is a quasi-uniform collection of partitions of H, there exists a positive constant K such that*

$$\|D^\alpha(f - P^N f)\|_H \leqq K(\bar{\rho})^{t-|\alpha|+\gamma} |f|_{t,\gamma,H}, \qquad 0 \leqq |\alpha| \leqq t,$$

for all $f \in C^{t,\gamma}(H)$, $0 \leqq \gamma \leqq 1$, $0 \leqq t \leqq k$, and all $\rho \in C$.

THEOREM 5.8. *Let C be a quasi-uniform collection of partitions of H.*

(i) *There exists a positive constant K such that*

$$\|D^\alpha(f - Q^N f)\|_H \leqq K(\bar{\rho})^{t-|\alpha|} \omega^{(t)}(\bar{\rho}; f), \qquad 0 \leqq |\alpha| \leqq t,$$

for all $f \in C_0(H) \cap C^t(H)$, $0 \leqq t \leqq k$, and all $\rho \in C$.

(ii) *There exists a positive constant K such that*

$$\|D^\alpha(f - Q^N f)\|_H \leqq K(\bar{\rho})^{t-|\alpha|+1}\left(\max_{|q|=t+1} \|D^q f\|_H\right), \qquad 0 \leqq |\alpha| \leqq t,$$

for all $f \in C_0(H) \cap S^{t+1,\infty}(H)$, $0 \leqq t \leqq k$, *and all* $\rho \in C$.

As a direct corollary of Theorem 5.8 we have the following theorem.

THEOREM 5.9. *If C is a quasi-uniform collection of partitions of H, there exists a positive constant K such that*

$$\|D^\alpha(f - Q^N f)\|_H \leqq K(\bar{\rho})^{t-|\alpha|+\gamma}|f|_{t,\gamma,H}, \qquad 0 \leqq |\alpha| \leqq t,$$

for all $f \in C_0(H) \cap C^{t,\gamma}(H)$, $0 \leqq \gamma \leqq 1$, $0 \leqq t \leqq k$, *and all* $\rho \in C$.

Combining Theorem 5.2 (ii) with the extension theorem of Hestenes [18] gives us the following theorem.

THEOREM 5.10. *Let S be a compact subset of* R^N *contained in the rectangular parallelepiped H and* $\rho \equiv \times_{i=1}^N \Delta_i$ *be a partition of H. There exists a positive constant K such that*

$$E(f; S_\rho^k) \equiv \inf\{\|f - s\|_S | s \in S_\rho^k\} \leqq K\omega(\bar{\rho}; f) \quad \textit{for all } f \in C(S),$$

where

$$S_\rho^k \equiv \bigotimes_{i=1}^N S_{\Delta_i, z(i)}^k.$$

Proof. By the extension theorem of Hestenes, there exists an extension $\tilde{f} \in C(H)$ of f such that $\omega(\bar{\rho}; \tilde{f}) = \omega(\bar{\rho}; f)$. The result follows from Theorem 5.2(ii) and the observation that $\|f - s\|_S \leqq \|\tilde{f} - s\|_H$ for all $s \in S_\rho^k$.

In a similar fashion we obtain the following two results by combining the Calderon extension theorem, cf. [8] and [23], with Theorem 5.7.

THEOREM 5.11. *Let* Ω *be a regular, bounded, open set in* R^N *such that* $\bar{\Omega} \subset \operatorname{int} H$, *a rectangular parallelepiped, and C be a quasi-uniform collection of partitions,* $\rho \equiv \times_{i=1}^N \Delta_i$, *of H. There exists a positive constant K such that*

$$E^\alpha(f; S_\rho^k) \equiv \inf\{\|D^\alpha(f - s)\|_\Omega | s \in S_\rho^k\} \leqq K(\bar{\rho})^{t-|\alpha|+\gamma}|f|_{t,\gamma,\Omega}, \qquad 0 \leqq |\alpha| \leqq t,$$

for all $f \in C^{t,\gamma}(\Omega)$, $0 \leqq \gamma \leqq 1$, $0 \leqq t \leqq k$, *and all* $\rho \in C$.

Using the Sobolev Theorem, cf. [8] and [23], we have the following theorem.

THEOREM 5.12. *Let* Ω *be a regular, bounded, open set of* R^N *such that* $\bar{\Omega} \subset \operatorname{int} H$, *a rectangular parallelepiped, and C be a quasi-uniform collection of partitions,* $\rho \equiv \times_{i=1}^N \Delta_i$, *of H.*

(i) *If* $m - N/p = t + \gamma$, *where t is a nonnegative integer and* $0 < \gamma < 1$, *then there is a positive constant K such that*

$$E^\alpha(f; S_\rho^k) \leqq K(\bar{\rho})^{t-|\alpha|+\gamma}\|f\|_{W^{m,p}(\Omega)}, \qquad 0 \leqq |\alpha| \leqq t,$$

for all $f \in W^{m,p}(\Omega)$, $1 < p < \infty$, $0 \leqq t \leqq k$, *and all* $\rho \in C$.

(ii) *If* $m - N/p = t + 1$, *where t is a nonnegative integer, then for each* $0 < \gamma < 1$ *there is a positive constant* K_γ *such that*

$$E^\alpha(j; S_\rho^k) < K_\gamma(\bar{\rho})^{t-|\alpha|+\gamma}\|f\|_{W^{m,p(\Omega)}}, \qquad 0 < |\alpha| \leqq t,$$

for all $f \in W^{m,p}(\Omega)$, $1 < p < \infty$, $0 \leqq t \leqq k$, *and all* $\rho \in C$.

6. Multivariate piecewise Lagrange interpolation. In this section we consider error estimates for multivariate, piecewise, Lagrange interpolation. Let $I \equiv [a, b]$, $-\infty < a < b < \infty$, and $\Delta : a = x_0 < x_1 < \cdots < x_m < x_{m+1} = b$, $N \geqq 1$, be a partition of I.

DEFINITION 6.1. For k a positive integer and Δ a partition of I, let $L^{(k)}(\Delta; I)$ be the set of all real-valued piecewise-polynomial functions $w(x)$ defined on I such that $w(x) \in C(I)$ and $w(x)$ is a polynomial of degree k on each subinterval $[x_i, x_{i+1}]$, $0 \leqq i \leqq m$, of I defined by Δ.

DEFINITION 6.2. Given any real-valued function $f \in C(I)$ and a subpartition of Δ of the form

$$\Delta^* : x_i = y_0(i) < y_1(i) < \cdots < y_{k-1}(i) < y_k(i) = x_{i+1}, \qquad 0 \leqq i \leqq m, \tag{6.1}$$

let its (unique) Δ^*-interpolate be the element f_{k,Δ^*} of $L^{(k)}(\Delta; I)$ such that $f(y_j(i)) = f_{k,\Delta^*}(y_j(i))$ for all $0 \leqq j \leqq k$, $0 \leqq i \leqq m$.

Following [3] we obtain the next theorem by using the Peano kernel theorem.

THEOREM 6.1. *Let Δ^* be a subpartition of Δ of the form* (6.1). *Then with* $s \equiv \min(t, k+1)$, *there exists a positive constant K such that*

$$\|D^j(f - f_{k,\Delta^*})\|_I \leqq K(\bar{\Delta})^{s-j-1/r}\|D^s f\|_{L^r(I)}, \qquad 0 \leqq j \leqq s-1,$$

for all $f \in K^{t,r}(I)$, where $t \geqq 1$, and f_{k,Δ^*} is the Δ^*-interpolate of f.

As in § 5, $\|D^j(f - f_{k,\Delta^*})\|_I$, $1 \leqq j \leqq s-1$, is interpreted as the uniform norm over $I - \Delta$.

From § 2 it is clear that it is important to know when the linear mappings $\mathscr{M} \equiv \mathscr{M}(k; \Delta^*) : C(I) \to C(I)$ defined by $\mathscr{M}(f) \equiv f_{k,\Delta^*}$ are uniformly bounded with respect to Δ^*. We remark that in the notation of § 5, $\mathscr{M}(1; \Delta) \equiv P_{\Delta,1,z}$ with $z = (1, \cdots, 1)$. Hence, in this section we shall concern ourselves with the cases of $k \geqq 2$.

Let $\bar{\Delta}_i^* \equiv \max_{0 \leqq j \leqq k}(y_{j+1}(i) - y_j(i))$ and $\underline{\Delta}_i^* \equiv \min_{0 \leqq j \leqq k}(y_{j+1}(i) - y_j(i))$, $0 \leqq i \leqq m$.

LEMMA 6.1. *If $k \geqq 2$ and C_τ is the collection of all subpartitions such that $(\bar{\Delta}_i^*/\underline{\Delta}_i^*) \leqq \tau$ for all $0 \leqq i \leqq m$, then $\|\mathscr{M}(k; \Delta^*)\| \leqq (k+1)^{k+1}\tau$.*

Proof. Note that in the interval $[x_i, x_{i+1}]$, $0 \leqq i \leqq m$, f_{k,Δ^*} is given by the polynomial $\sum_{j=0}^k f(y_j(i)) l_j(x)$, where

$$l_j(x) \equiv \prod_{\substack{s=0 \\ s \neq j}}^k \frac{x - y_s(i)}{y_j(i) - y_s(i)}.$$

Thus

$$\|f_{k,\Delta^*}\|_{[x_i, x_{i+1}]} \leqq \|f\|_I \sum_{j=0}^k \|l_i(x)\|_{[x_i, x_{i+1}]} \leqq \|f\|_I (k+1)^{k+1}\tau^k.$$

COROLLARY. *If $\{\Delta(n)\}_{n=1}^\infty$ is any sequence of partitions of I such that $\bar{\Delta}(n) \to 0$ as $n \to \infty$, $\{\Delta^*(n)\}_{n=1}^\infty$ is a corresponding sequence of subpartitions such that there exists a positive constant τ such that $(\bar{\Delta}_i^*(n)/\underline{\Delta}_i^*(n)) \leqq \tau$ for all $0 \leqq i \leqq m(n)$ and all $n \geqq 1$, $f(x) \in C(I)$, and $f_{k,\Delta^*(n)}$ is the $\Delta^*(n)$-interpolate of $f(x)$, then $\|f - f_{k,\Delta^*(n)}\|_I \to 0$ as $n \to \infty$.*

Proof. This follows directly from Theorem 6.1 and Lemma 6.1.

We now proceed to multivariate piecewise Lagrange interpolation. As before, we consider rectangular parallelepipeds of the form

$$H \equiv \mathop{X}_{j=1}^{N} [a_j, b_j], \qquad -\infty < a_j < b_j < \infty, \quad 1 \leqq j \leqq N.$$

Let Δ_j and Δ_j^* denote a partition and subpartition of $[a_j, b_j]$, $1 \leqq j \leqq N$, respectively. Let

$$\rho \equiv \mathop{X}_{j=1}^{N} \Delta_j \quad \text{and} \quad \rho^* \equiv \mathop{X}_{j=1}^{N} \Delta_j^*.$$

Given $f \in C(H)$, for each $1 \leqq j \leqq N$, define $\mathcal{M}_j(f) \equiv \mathcal{M}_j(k; \Delta_j^*)(f)$ as the function obtained by applying the mapping $\mathcal{M}(k; \Delta_j^*)$ to $f(x)$, viewed as a function of x_j while holding the variables x_k, $k \neq j$, fixed, and $\mathcal{M}^N \equiv \mathcal{M}^N(k; \rho^*) \equiv \mathcal{M}_N(k; \Delta_N^*) \cdots \mathcal{M}_1(k; \Delta^*)$. For each positive integer t, let $K^{t,r,\infty}(H)$ denote the set of all real-valued functions f on H such that for each $1 \leqq j \leqq N$, $D_j^t f \in L^r(a_j, b_j)$, almost everywhere in H_j, and $\|D_j^t f\|_{L^r(a_j,b_j)} \in L^\infty(H_j)$. For each $1 \leqq j \leqq N$, we denote the combined norm by $\|\cdot\|_{L_j^r \times L^\infty}$. Combining Theorems 2.1 and 6.1 and Lemma 6.1 gives us the following theorem.

THEOREM 6.2. *Let C_τ be the collection of all subpartitions ρ^* of H, such that $((\bar{\Delta}_j)_i^*/(\Delta_j)_i^*) \leqq \tau$ for all $0 \leqq i \leqq m_j$ and $1 \leqq j \leqq N$.*

(i) $\|\mathcal{M}^N(k; \rho^*)\| \leqq (k+1)^{(k+1)N} \tau^{kN}$ *for all* $\rho^* \in C_\tau$.

(ii) $\|f - \mathcal{M}^N f\|_H \leqq \sum_{j=1}^N (k+1)^{(k+1)(N-j)} \tau^{k(N-j)} (\bar{\Delta}_j)^{s-1/r} \|D_j^s f\|_{L_j^r \times L^\infty}$, *where* $s \equiv \min(t, 2k)$, *for all* $f \in K^{t,r,\infty}(H)$, $1 \leqq t$, *and* $\rho^* \in C_\tau$.

COROLLARY. *If $\{\rho(n)\}_{n=1}^\infty$ is any sequence of partitions of H such that*

$$\max_{1 \leqq j \leqq N} \bar{\Delta}_j(n) \to 0 \text{ as } n \to \infty, \ \{\rho^*(n)\}_{n=1}^\infty$$

is a corresponding sequence of subpartitions such that there exists a positive constant τ such that $((\bar{\Delta}_j)_i^*(n)/(\Delta_j)_i^*(n)) \leqq \tau$ for all $0 \leqq i \leqq m(j,n)$, $1 \leqq j \leqq N$, $n \geqq 1$, and $f(x) \in C(H)$, then $\|f - \mathcal{M}^N(n) f\|_H \to 0$ as $n \to \infty$.

In a fashion similar to the proofs of Lemmas 5.1, 5.2 and 5.3, we may prove the following lemma.

LEMMA 6.2. (i) *If $f \in C(H)$, $\mathcal{M}_i \mathcal{M}_j f = \mathcal{M}_j \mathcal{M}_i f$ for all $1 \leqq i, j \leqq N$.*

(ii) *If $f \in C(H)$ is such that $D_i f \in C(H)$ for some $1 \leqq i \leqq N$, then $D_i(\mathcal{M}_j f) = \mathcal{M}_j(D_i f)$ for all $1 \leqq j \leqq N$ such that $j \neq i$.*

Combining Theorems 6.1 and 6.2 with Lemma 6.1 gives us the following theorem.

THEOREM 6.3. *Let C_τ be the collection of all subpartitions ρ^* of H, such that $((\bar{\Delta}_j)_i^*/(\Delta_j)_i^*) \leqq \tau$ for all $0 \leqq i \leqq m_j$ and $1 \leqq j \leqq N$.*

(i) *There exists a positive constant K such that*

$$\|D_l^j(f - \mathcal{M}^N f)\|_H \leqq K \sum_{j=1}^N (\bar{\Delta}_j)^{s-j-1/r} \|D_j^s f\|_{L_j^r \times L^\infty}, \quad 0 \leqq j \leqq s-1, \ 1 \leqq l \leqq N,$$

where $s \equiv \min(t, 2k)$, for all $f \in K^{t,r,\infty}(H)$, $1 \leqq t$, and $\rho^* \in C_\tau$.

(ii) *There exists a positive constant* K *such that*

$$\|D^\alpha(f - \mathscr{M}^N f)\|_H \leqq K(\bar{\rho})^{s-|\alpha|}(\max_{|q|=s} \|D^q f\|_H), \quad 0 \leqq |\alpha| \leqq s-1,$$

where $s \equiv \min(t, 2k)$, *for all* $f \in S^{t,\infty}(H)$, $1 \leqq t$, *and all* $\rho^* \in C_\tau$.

As in § 5, $\|D^\alpha(f - \mathscr{M}^N f)\|_H$, $1 \leqq |\alpha|$, is interpreted as the uniform norm over the interior of the N-cells defined by the partition ρ.

7. Multivariate cubic spline interpolation. In this section we recall a cubic spline interpolation scheme introduced by deBoor [13]. An analogous scheme is introduced for multivariate cubic spline interpolation in rectangular parallelepipeds, and asymptotic bounds are obtained for the interpolation error.

In this section, let $I \equiv [a, b]$, $-\infty < a < b < \infty$, $\Delta: a = x_0 < x_1 < \cdots < x_m < x_{m+1} = b$, $m \geqq 1$, be a partition of I, $\tilde{\Delta}: a = x_0 < x_2 < x_3 < x_4 < \cdots < x_{m-3} < x_{m-2} < x_{m-1} < x_{m+1} = b$ be the associated partition obtained from Δ by omitting the points x_1 and x_m, $S^3_{\tilde{\Delta}} \equiv S^3_{\tilde{\Delta}}(I)$ be the set of all polynomial spline functions of degree 3 on I with respect to $\tilde{\Delta}$, i.e., $s(x) \in S^3_{\tilde{\Delta}}$ if and only if $s(x)$ is a cubic polynomial on each subinterval defined by $\tilde{\Delta}$ and $s(x) \in C^2(I)$, and $\bar{\Delta} \equiv \max_{0 \leqq i \leqq m}(x_{i+1} - x_i)$, $\underline{\Delta} \equiv \min_{0 \leqq i \leqq m}(x_{i+1} - x_i)$.

Following deBoor [13], we define the linear interpolation mapping $T_\Delta: C(I) \to S^3_{\tilde{\Delta}}$ by $T_\Delta f = s_f \in S^3_{\tilde{\Delta}}$, where

$$s_f(x_i) = f(x_i), \qquad 0 \leqq i \leqq m+1, \tag{7.1}$$

and recall the following fundamental result.

THEOREM 7.1. (i) *T_Δ is a well-defined linear mapping of $C(I)$ into $S^3_{\tilde{\Delta}}$.*

(ii) $\|T_\Delta\| \leqq 1 + \frac{5}{2}(\bar{\Delta}/\underline{\Delta})^2$.

(iii) $\|f - T_\Delta f\|_I \leqq (1 + \frac{5}{4}(\bar{\Delta}/\underline{\Delta})^2)\omega(\bar{\Delta}; f)$ *for all* $f \in C(I)$.

(iv) *If C is a quasi-uniform collection of partitions Δ of I, then there exists a positive constant K such that*

$$\|D^j(f - T_\Delta f)\|_I \leqq K(\bar{\Delta})^{4-j}, \qquad 0 \leqq j \leqq 2, \tag{7.2}$$

for all $\Delta \in C$, *for all* $f \in K^{4,\infty}(I)$.

We now proceed to a multivariate analogue of the above interpolation scheme. As before, we consider rectangular parallelepipeds of the form

$$H \equiv \bigtimes_{i=1}^{N} [a_i, b_i], \ -\infty < a_i < b_i < \infty, \qquad 1 \leqq i \leqq N.$$

Let Δ_i and $\tilde{\Delta}_i$ denote a partition and associated partition of $[a_i, b_i]$, $1 \leqq i \leqq N$, respectively. Let $\rho \equiv \bigtimes_{i=1}^N \Delta_i$, $\tilde{\rho} \equiv \bigtimes_{i=1}^N \tilde{\Delta}_i$, $\bar{\rho} \equiv \max_{1 \leqq i \leqq N} \bar{\Delta}_i$, and $\underline{\rho} \equiv \min_{1 \leqq i \leqq N} \underline{\Delta}_i$. Given $f \in C(H)$, for each $1 \leqq i \leqq N$ define $T_{\Delta_i}(f)$ as the function obtained by applying the mapping T_{Δ_i} to $f(x)$, viewed as a function of x_i while holding the variables x_j, $j \neq i$, fixed, and $T^N_\rho \equiv T_{\Delta_N} \cdots T_{\Delta_1}$.

Combining Theorems 2.1 and 7.1, we have the next theorem.

THEOREM 7.2. (i) $\|T^N_\rho\| \leqq [1 + \frac{5}{2}(\bar{\rho}/\underline{\rho})^2]^N$.

(ii) $\|f - T^N_\rho f\|_H \leqq \sum_{i=1}^N \omega_i(\bar{\Delta}_i; f)[1 + \frac{5}{4}(\bar{\Delta}_i/\underline{\Delta}_i)^2][1 + \frac{5}{2}(\bar{\rho}/\underline{\rho})^2]^{N-i}$ *for all* $f \in C(H)$.

COROLLARY. *If $\{\rho(n)\}_{n=1}^{\infty}$ is any quasi-uniform sequence of partitions of H such that $\max_{1 \leqq j \leqq N} \bar{\Delta}_j(n) \to 0$ as $n \to \infty$, then $\|f - T^N_{\rho(n)}f\|_H \to 0$ as $n \to \infty$ for all $f \in C(H)$.*

In a fashion similar to the proofs of Lemmas 5.1, 5.2 and 5.3, we may show the following lemma.

LEMMA 7.1. (i) *If $f \in C(H)$, $T_{\Delta_i}T_{\Delta_j}f = T_{\Delta_j}T_{\Delta_i}f$ for all $1 \leqq i, j \leqq N$.*

(ii) *If $f \in C(H)$ is such that $D_i f \in C(H)$ for some $1 \leqq i \leqq N$, then $D_i(T_{\Delta_j}f) = T_{\Delta_j}(D_i f)$ for all $1 \leqq j \leqq N$ such that $j \neq i$.*

Combining Theorems 7.1 and 7.2 with Lemma 7.1, we obtain the following theorem.

THEOREM 7.3. *Let C be a quasi-uniform collection of partitions of H.*

(i) *There exists a positive constant K such that*

$$\|D_l^j(f - T^N_\rho f)\|_H < K \sum_{i=1}^{N} (\bar{\Delta}_i)^{4-j}, \qquad 0 \leqq j \leqq 2, \tag{7.3}$$

$1 \leqq l < N$, for all $f \in K^{4,\infty}(H)$ and all $\rho \in C$.

(ii) *There exists a positive constant K such that*

$$\|D^\alpha(f - T^N_\rho f)\|_H \leqq K(\bar{\rho})^{4-|\alpha|}, \qquad 0 \leqq |\alpha| \leqq 2, \tag{7.4}$$

for all $f \in S^{4,\infty}(H)$ and all $\rho \in C$.

8. *h*-Asymptotically optimal approximation schemes. In this section, we discuss the application of the approximation theory results of § 5 to the study of the truncation or discretization error for abstract h-asymptotically optimal approximation schemes, cf. Definition 8.1. In special cases, this study yields rigorous error bounds for the Rayleigh–Ritz–Galerkin method for a class of complex-monotone operator equations, cf. [1], [2], [7] and [11], including a wide class of boundary value problems for nonlinear elliptic partial differential equations, cf. [11], and for a wide class of eigenvalue problems for linear elliptic partial differential equations, cf. [16]. Related error bounds for the Rayleigh–Ritz–Galerkin method are given in [1], [2], [3], [7], [9], [10], [11], [15], [16], [17], [22], [25] and [26].

DEFINITION 8.1. Let $u \in B$, a real or complex Banach space, F be any family of finite-dimensional subspaces S of B such that $u \in \overline{\bigcup_{S \in F} S}$, and $\mathscr{M}_F$ be any family $\{\mathscr{M}_S\}_{S \in F}$ of mappings of u into B such that $\mathscr{M}_S u \in S$ for all $S \in F$. If h is a monotonically increasing continuous function on $R^+ \cup 0$ such that $h(0) = 0$ and $\lim_{x \to \infty} h(x) = \infty$, then $\mathscr{M}_F$ is said to be an *h-asymptotically optimal approximation scheme* if and only if there exists a positive constant K such that

$$\|u - \mathscr{M}_S u\|_B \leqq h\{K \inf_{y \in S} \|u - y\|_B\} \equiv h\{KE_B(u; S)\} \tag{8.1}$$

for all $S \in F$. If $h(x) = kx$, for some positive constant k, $\mathscr{M}_F$ is said to be an *asymptotically optimal approximation scheme.*

If $B = W^{m,p}(\Omega)$ for a regular, bounded, open set $\Omega \subset R^N$ and F is a special family of finite-dimensional subspaces of multivariate spline functions, and we know something about the smoothness of u, then using the results of § 5 we can bound the quantity $E_{W^{m,p}(\Omega)}(u; S)$ which appears on the right-hand side of (8.1).

THEOREM 8.1. *Let $\Omega \subset R^N$ be a regular, bounded, open set such that $\bar{\Omega} \subset \operatorname{int} H$, a rectangular parallelepiped, C be a quasi-uniform collection of partitions $\rho \equiv X_{i=1}^N \Delta_i$ of H, $u \in W^{m,p}(\Omega)$, $F \equiv \{S_\rho^k | S_\rho^k \subset W^{m,p}(\Omega)$ and $\rho \in C\}$, and $\mathscr{M}_F$ be an h-asymptotically optimal approximation scheme in $W^{m,p}(\Omega)$.*

(i) *There exists a positive constant K such that*

$$\|u - \mathscr{M}_{S_\rho^k} u\|_{W^{m,p}(\Omega)} \leqq h\{K(\bar{\rho})^{t-m+\gamma}|u|_{t,\gamma,\Omega}\} \tag{8.2}$$

for all $u \in C^{t,\gamma}(\Omega)$, $0 \leqq \gamma \leqq 1$, where $0 \leqq m \leqq t$ and all $S_\rho^k \in F$ such that $t \leqq k$.

(ii) *If $s - N/q = t + \gamma$, where $0 \leqq m \leqq t$ is an integer and $0 < \gamma < 1$, then there is a positive constant K such that*

$$\|u - \mathscr{M}_{S_\rho^k} u\|_{W^{m,p}(\Omega)} \leqq h\{K(\bar{\rho})^{t-m+\gamma}\}\|u\|_{W^{s,q}(\Omega)}\} \tag{8.3}$$

for all $u \in W^{s,q}(\Omega)$, $1 < q < \infty$, and all $S_\rho^k \in F$ such that $t \leqq k$.

(iii) *If $s - N/q = t + 1$, where $0 \leqq m \leqq t$ is an integer, then for each $0 < \gamma < 1$ there is a positive constant K_γ such that*

$$\|u - \mathscr{M}_{S_\rho^k} u\|_{W^{m,p}(\Omega)} \leqq h\{K_\gamma(\bar{\rho})^{t-m+\gamma}\|u\|_{W^{s,q}(\Omega)}\} \tag{8.4}$$

for all $u \in W^{s,q}(\Omega)$, $1 < q < \infty$, and all $S_\rho^k \in F$ such that $t \leqq k$.

If $B \equiv W_0^{1,p}(\Omega)$, H is a rectangular parallelepiped, F is a special family of finite-dimensional subspaces of multivariate spline functions, and we know something about the smoothness of u, then the results of § 5 can again be used to bound $E_{W_0^{1,p}(H)}(u; S)$.

THEOREM 8.2. *Let $H \subset R^N$ be a rectangular parallelepiped, C be a quasi-uniform collection of partitions $\rho \equiv \bigtimes_{i=1}^N \Delta_i$ of H, $u \in W_0^{1,p}(H)$, $F \equiv \{S_\rho^k | S_\rho^k \subset W_0^{1,p}(H)$ and $\rho \in C\}$, and $\mathscr{M}_F$ be an h-asymptotically optimal approximation scheme in $W_0^{1,p}(H)$.*

(i) *There exists a positive constant K such that*

$$\|u - \mathscr{M}_{S_\rho^k} u\|_{W_0^{1,p}(H)} \leqq h\{K(\bar{\rho})^{t-1+\gamma}|u|_{t,\gamma,H}\} \tag{8.5}$$

for all $u \in C^{t,\gamma}(H)$, $0 \leqq \gamma \leqq 1$, where $1 \leqq t$, and all $S_\rho^k \in F$ such that $t \leqq k$.

(ii) *If $s - N/q = t + \gamma$, where $1 \leqq t$ is an integer and $0 < \gamma < 1$, then there is a positive constant K such that*

$$\|u - \mathscr{M}_{S_\rho^k} u\|_{W_0^{1,p}(H)} \leqq h\{K(\bar{\rho})^{t-m+\gamma}\|u\|_{W^{s,q}(H)} \tag{8.6}$$

for all $u \in W^{s,q}(H)$, $1 < q < \infty$, and all $S_\rho^k \in F$ such that $t \leqq k$.

(iii) *If $s - N/q = t + 1$, where $1 \leqq t$ is an integer, then for each $0 < \gamma < 1$, there is a positive constant K_γ such that*

$$\|u - \mathscr{M}_{S_\rho^k} u\|_{W_0^{1,p}(H)} \leqq h\{K_\gamma(\bar{\rho})^{t-1+\gamma}\|u\|_{W^{s,q}(H)}\} \tag{8.7}$$

for all $u \in W^{s,q}(H)$, $1 < q < \infty$, and all $S_\rho^k \in F$ such that $t \leqq k$.

As an example of the possible applications of Theorems 8.1 and 8.2, we discuss a class of difference methods which are applicable to the Neumann problem for the Poisson equation in multidimensional regions. The difference equations are obtained from the variational principle of the boundary value problem by using the Ritz method in conjunction with special finite dimensional subspaces of multivariate spline functions. For a discussion of the Ritz method in conjunction with finite-dimensional subspaces of functions which are piecewise linear over a triangulation see [17].

The problem under consideration is to find a solution $u(x)$ of the equation

$$\Delta u(x) \equiv \sum_{i=1}^{N} D_i^2 u(x) = f(x), \qquad x \in \Omega \subset R^N, \tag{8.8}$$

where Ω is a regular, bounded, open set, subject to the boundary condition

$$\frac{\partial u}{\partial n}(x) = g(x), \qquad x \in \partial\Omega, \tag{8.9}$$

where n refers to the exterior normal to the boundary $\partial\Omega$ of the region Ω.

In order that a solution exist, it is necessary that the inhomogeneous data, f and g, satisfy the compatibility equation

$$\oint_{\partial\Omega} g\, ds = \int_{\Omega} f\, dx. \tag{8.10}$$

Under appropriate smoothness conditions on $\partial\Omega$, f, and g, a solution will exist and is unique to within an additive constant. We make the solution unique by requiring that

$$\int_{\Omega} u\, dx = 0. \tag{8.11}$$

Let the functional $D[\varphi, \psi]$ be defined by

$$D[\varphi, \psi] \equiv \int_{\Omega} \left\{ \sum_{i=1}^{N} (D_i\varphi)(D_i\psi) \right\} dx. \tag{8.12}$$

Then the solution u of (7.7)–(7.8), modulo an additive constant, minimizes the functional

$$F[\varphi] = \tfrac{1}{2}D[\varphi, \varphi] + \int_{\Omega} f\varphi\, dx - \oint_{\partial\Omega} g\varphi\, ds \tag{8.13}$$

over the space $\Phi \subset W^{1,2}(\Omega)$, all real-valued functions, $\varphi \in C(\bar{\Omega})$, which have piecewise continuous first partial derivatives.

If θ denotes any finite-dimensional subspace of Φ, the functional $F[\varphi]$ has a minimum over θ which is assumed at u_θ. The minimizing function u_θ is unique up to an additive constant and is made unique by requiring that

$$\int_{\Omega} u_\theta\, dx = 0. \tag{8.14}$$

If for every finite-dimensional subspace $\theta \subset \Phi$, we define the mapping $\mathscr{M}_\theta : u \to u_\theta$, then we have the following important result of Friedrichs and Keller [17].

THEOREM 8.3. *Let $u \in \Phi \subset W^{1,2}(\Omega)$ be the solution of* (8.8)–(8.9), *F be the family of all finite-dimensional subspaces θ of Φ, and F be the family of mappings defined by $\mathcal{M}_\theta : u \to u_\theta$ for all $\theta \in F$. Then $\mathcal{M}_F$ is an asymptotically optimal approximation scheme in $W^{1,2}(\Omega)$.*

Combining Theorem 8.1(i) and 8.3, we obtain the following theorem.

THEOREM 8.4. *Let $\Omega \subset R^N$ be a regular, bounded, open set such that $\bar{\Omega} \subset \operatorname{int} H$, a rectangular parallelepiped, C be a quasi-uniform collection of partitions $\rho \equiv \times_{i=1}^{N} \Delta_i$ of H, u be the solution of* (8.8)–(8.9), *$F \equiv \{S_\rho^k | \rho \in C\}$, and $\mathcal{M}_F \equiv \{\mathcal{M}_\theta\}_{\theta \in F}$, where $\mathcal{M}_\theta$ maps u into the Ritz approximation u_θ in θ. Then, if $u \in C^{t,\gamma}(\Omega)$, where $1 \leqq t$ and $0 \leqq \gamma \leqq 1$, there exists a constant K such that*

$$\|u - u_\theta\|_{W^{1,2}(\Omega)} \leqq K(\bar{\rho})^{t-1+\gamma}|u|_{t,\gamma,\Omega} \tag{8.15}$$

for all $S_\rho^k \in F$ such that $t \leqq k$.

We remark that if $u \in C^{3,1}(\Omega)$, then the above results yields $\|u - u_\theta\|_{W^{1,2}(\Omega)} \leqq K(\bar{\rho})^3|u|_{3,1,\Omega}$ for all $S_\rho^3 \in F$, and if $u \in C^{2,1}(\Omega)$, then it yields $\|u - u_\theta\|_{W^{1,2}(\Omega)} \leqq K(\bar{\rho})^2|u|_{2,1,\Omega}$ for all $S_\rho^2 \in F$. Thus, these approximation schemes compare favorably with those of Bramble and Hubbard [5] for the Neumann problem for the Poisson equation. In fact, their finite difference approximation yields an approximation solution, u_h, with the properties that $\|u - u_h\|_\infty = O(h^2|\ln h|)$, as $h \to 0$, if $u \in C^4(\bar{\Omega})$, and $\|u - u_h\|_\infty = O(h|\ln h|)$, as $h \to 0$, if $u \in C^3(\bar{\Omega})$, where $\|\cdot\|_\infty$ denotes the maximum over the mesh points. Furthermore, if the solution u is sufficiently differentiable, the Ritz method described in Theorem 8.4 will be of "high-order accuracy," if the spline subspace S_ρ^k is chosen with k sufficiently large, whereas no "high-order accurate" finite difference approximations are known.

For further possible applications of Theorems 8.1 and 8.2 the reader is referred to [1], [2], [3], [7], [9], [10], [11], [15], [16], [17], [22], [25] and [26].

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